Convergence Theorems to Solution of Variation Inequalities
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Abstract
In this paper, we introduce a new proximal point schemes of metric projection and wider mappings and we discuss the (strong/weak) convergence for these proximal point schemes in Hilbert space.

INDEXING TERMS/KEYWORDS
Hilbert spaces, maximal monotone, strongly convergence, wider mappings

SUBJECT CLASSIFICATION
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1. INTRODUCTION

Let $X$ be a real Hilbert space and $A$ be multivalued mapping with domain $D(A)$ and range $\text{R}(A)$. The mapping $A$ is called monotone mapping if

$$< x_1 - x_2, y_1 - y_2 > \geq 0 \quad \forall x_i \in D(A), \quad y_i \in A(x_i), \quad i = 1, 2.$$ 

Also, any mapping $A$ is called maximal monotone mapping of $\mathcal{A}$ if the graph $G(A)$ of $A$ is not properly contained in the graph of any other monotone mapping. The Monotone mappings play a crucial role in modern nonlinear analysis and optimization, see the books ([1]—[5]). And the convergence of the proximal point schemes are studied by many researchers see ([6]—[14]).

Let $C$ nonempty subset of $X$, and we use $x_n \to z$ $(x_n \rightharpoonup z)$ to strong (weak, respectively) convergence of $x_n$ to $z$. The metric projection of $X$ onto $C$:

For each $x \in X$ there exist a unique point $z \in C$ such that

$$||x - z|| = \inf ||x - y||; \quad y \in C$$

(1)

Such that denote by $P_C(x)$ and $P_C$ is called the metric projection of $X$ onto $C$. It is known that

$$\langle x - P_C(x), y - P_C(x) \rangle \leq 0 \quad \text{for all } \quad y \in C$$

(2)

The metric projection operator plays an important role in fixed point theory, optimization theory, variational inequality problems and games theory. We recall some definitions and lemmas.

We recall some lemmas which will used in the proofs.

Lemma (1.1) : [14]

Let $C$ be a nonempty convex closed subset of real Hilbert space $X$ and $T$ is non-expansive multivalued mapping such that $Fix(T) \neq \emptyset$. Then $T$ is demi-closed.

Lemma (1.2) : [7]

If $\langle \alpha_n \rangle$ be a sequence of non-negative real number such that:

$$\alpha_{n+1} \leq (1 - \gamma_n) \alpha_n + S_n, \quad n \geq 0$$

Where $\langle \gamma_n \rangle$ is a sequence in $(0,1)$ and $\langle S_n \rangle$ be a sequence in $\mathbb{R}$ such that:

$$\sum_{n=0}^{\infty} \gamma_n = \infty \quad \text{and} \quad \lim_{n \to \infty} \sup \frac{S_n}{\gamma_n} \leq 0 \quad \text{or} \quad \sum_{n=1}^{\infty} |S_n| < \infty.$$

Then $\alpha_n \to 0$
Lemma (1.3) : \cite{[15]}

If \( \{a_n\} \) be a sequence nonnegative real numbers such that:
\[
a_{n+1} \leq (1 - \gamma) a_n + \gamma_n S_n + \beta_n, \quad n \geq 0
\]

Where \( \{\gamma_n\}, \{\beta_n\} \) and \( \{S_n\} \) are satisfies the following:

1. \( \gamma_n \in [0, 1] \) ; \( \sum_{n=1}^{\infty} \gamma_n = \infty \)
2. \( \lim_{n \to \infty} \sup S_n \leq 0 \) or \( \sum_{n=1}^{\infty} |\gamma_n S_n| < \infty \)
3. \( \beta_n \geq 0 \) for each \( n \geq 0 \) such that \( \sum_{n=0}^{\infty} \beta_n < \infty \). Then \( \lim_{n \to \infty} a_n = 0 \).

Definition (1.4) : \cite{[16]}

Let \( X \) be a normed space and \( C \) be a nonempty subset of \( X \), then \( A \) mapping \( T : C \to C \) is called \( szl \)-widering if for each \( s, l \in (0,1) \) then there exists \( z > 0 \) such that the following equation holds:
\[
\|Tx - Ty\| \leq (1 - s)\|x - y\|^2 + l\|y - Ty\| + z\|x - Tx, y - Ty\|, \text{for each } x, y \in C
\] (3)

The concept of \( szl \)-widering mapping is independent of concepts non-expansive.

Lemma (1.5) : \cite{[16]}

Let \( X \) be a closed convex subset of Hilbert space \( X \) and \( T \) is \( szl \)-widering. Then it is demi-closed, i.e., if there exists \( \{x_n\} \) sequence in \( C \) such that \( x_n \to p \) and \( \|Tx_n - x_n\| \to 0 \) then \( p \in Fix(p) \).

In this paper, we give a new proximal point schemes of \( \alpha \)-inverse strongly monotone and \( szl \)-widering mappings, also, we prove that these proximal point schemes converge weakly and strongly to the common fixed point.

2. MAIN RESULTS

Theorem (2.1) :

Let \( \{f_n\} \) be a bounded sequence of \( szl \)-widering mappings and \( \{T_n\} \) be a sequence of non-expansive mapping. Define the proximal point scheme \( \{x_n\} \) as:
\[
x_{n+1} = a_n f_n x_n + b_n T_n x_n + c_n P_C T_n y_n
\]
\[
y_n = (1 - c_n) x_n + c_n P_C (x_n - w_n h x_n)
\]

where \( \{a_n\}, \{b_n\}, \{c_n\} \) are sequences in \( (0,1) \) converges to 0 such that \( a_n \geq c_n \) but \( \{w_n\} \) be a sequence in \( (1,\infty) \) not converges to 0 and satisfy \( w_n < 2a \). If

i. \( \text{VI}(C, h) \cap (\bigcap_{n=1}^{\infty} Fix(T_n)) \cap (\bigcap_{n=1}^{\infty} Fix(f_n)) \neq \emptyset \)

ii. \( a_n + b_n + c_n = 1 \)

Then the proximal point scheme \( \{x_n\} \) has converges weakly to an asymptotic common fixed point of \( T_n, \forall n \in N \)

Proof:

Let \( p \in \text{VI}(C, h) \cap (\bigcap_{n=1}^{\infty} Fix(T_n)) \cap (\bigcap_{n=1}^{\infty} Fix(f_n)) \),

Since \( p \in \text{VI}(C, h) \implies P_C (p - w_n h p) = p \)
\[ \|y_n - p\|^2 = \| (1 - c_n) x_n + c_n P_c (x_n - w_n h x_n) - p \|^2 \\
\leq (1 - c_n) \| x_n - p \|^2 + c_n \| P_c (x_n - w_n h x_n) - P_c (p - w_n h p) \|^2 \\
\leq (1 - c_n) \| x_n - p \|^2 + c_n \| x_n - w_n h x_n - (p - w_n h p) \|^2 \\
= (1 - c_n) \| x_n - p \|^2 + c_n \left( \| x_n - p \|^2 - 2 w_n \langle x_n - p, h x_n - h p \rangle \right) \\
+ w_n^2 \| h x_n - h p \|^2 \] 

But \( h \) is \( \alpha \) inverse strongly monotone mapping. Therefore,

\[ \|y_n - p\|^2 \leq \| x_n - p \|^2 - 2 w_n c_n \alpha \| h x_n - h p \|^2 + w_n^2 c_n \| h x_n - h p \|^2 \]

\[ \|y_n - p\|^2 \leq \| x_n - p \|^2 \]

Now, by definition of \( szl \) wider than we get, for each \( s_n, l_n \in (0,1) \) then there exist \( z_n > 0 \) (shortly we write them \( s, z \) and \( l \) respectively) such that

\[ \|x_{n+1} - p\|^2 \leq a_n \| f_n(x_n) - p \|^2 + b_n \| T_n x_n - p \|^2 + c_n \| P_c T_n x_n - p \|^2 \\
\leq a_n \| x_n - p \|^2 + b_n \| x_n - p \|^2 + c_n \| x_n - p \|^2 + c_n \left( \| x_n - p \|^2 - 2 w_n \langle x_n - p, h x_n - h p \rangle \right) \\
\leq a_n \| x_n - p \|^2 + b_n \| x_n - p \|^2 + c_n \| x_n - p \|^2 \leq a_n \| x_n - p \|^2 \]

Since, \( \langle x_n \rangle \) is bounded sequence then there exists a subsequence \( \langle x_{nk} \rangle \) of \( \langle x_n \rangle \) such that \( x_{nk} \rightarrow z \)

\[ \|x_n - T_n x_n\| \leq a_{n-1} \| x_{n-1} - T_{n-1} x_{n-1} - T_n x_n \| + b_{n-1} \| T_{n-1} x_{n-1} - T_n x_n \| + c_{n-1} \| P_c T_{n-1} x_{n-1} - T_n x_n \| \\
\|x_n - T_n x_n\| \rightarrow 0 \quad \text{as} \quad n \rightarrow \infty \]

Then the proximal point scheme \( \langle x_n \rangle \) has converges weakly to an asymptotic common fixed point of \( T_n \), \( \forall n \in N \). ■

**Theorem (2.2)**

If \( \langle f_n \rangle \) be a sequence of non-expansive mappings, \( \langle T_n \rangle \) be a sequence of \( szl \) wider mappings. Define the following the proximal point scheme as follows:

\[ y_n = b_n x_n + (1 - b_n) P_c (x_n - \lambda_n h x_n) \]

\[ x_{n+1} = a_n f_n x_n + (1 - a_n) T_n f_n y_n \]

where \( \langle a_n \rangle, \langle b_n \rangle \) are any sequences in \( (0, 1) \) converges to \( 0 \) and \( \langle \lambda_n \rangle \) be a sequence in \( (0, \infty) \) such that \( \lambda_n \leq 2 \alpha \) and

i. \( a_n + b_n = 1 \)

ii. \( VI(C, h) \cap (\cap_{n=1}^{\infty} Fix(T_n)) \cap (\cap_{n=1}^{\infty} Fix(f_n)) \neq \emptyset \)

Then the proximal point scheme \( \langle x_n \rangle \) converges weakly to an asymptotic common fixed point of \( f_n \), \( \forall n \in N \).

**Proof**

Let \( p \in VI(C, h) \cap (\cap_{n=1}^{\infty} Fix(T_n)) \cap (\cap_{n=1}^{\infty} Fix(f_n)) \)

\[ \|y_n - p\|^2 \leq b_n \| x_n - p \|^2 + (1 - b_n) \| P_c (x_n - \lambda_n h x_n) - p \|^2 \\
\leq b_n \| x_n - p \|^2 + (1 - b_n) \| x_n - \lambda_n h x_n - (p - \lambda_n h p) \|^2 \\
\leq b_n \| x_n - p \|^2 + (1 - b_n) \| x_n - p \|^2 + \lambda_n (1 - b_n) \| h x_n - h p \|^2 \\
- 2 \lambda_n (1 - b_n) \langle x_n - p, h x_n - h p \rangle \]

\[ \|y_n - p\|^2 \leq b_n \| x_n - p \|^2 + (1 - b_n) \| x_n - p \|^2 + \lambda_n (1 - b_n) \| h x_n - h p \|^2 \\
- 2 (1 - b_n) \lambda_n \alpha \| h x_n - h p \|^2 \]

\[ \|y_n - p\|^2 \leq \| x_n - p \|^2 + \lambda_n (\lambda_n - 2\alpha) (1 - b_n) \| h x_n - h p \|^2 \]

\[ \|y_n - p\|^2 \leq \| x_n - p \|^2 \]
Now, by definition of \( szl \)– widering then we get, for each \( s_n, l_n \in (0,1) \) then there exist \( z_n > 0 \) (shortly we write them \( s, z \) and \( l \) respectively) such that

\[
\|x_{n+1} - p\|^2 \leq a_n \|f_n x_n - p\|^2 + (1 - a_n) \|T_n f_n y_n - p\|^2
\]

\[
\leq a_n \|x_n - p\|^2 + (1 - a_n) (1 - s) \|f_n y_n - p\|^2 + l \|p - T_n p\| \|f_n y_n - T_n f_n y_n - (p - T_n p)\|
\]

\[
\leq a_n + z \langle f_n y_n - T_n f_n y_n, p - T_n p \rangle
\]

\[
\|x_{n+1} - p\|^2 \leq a_n \|x_n - p\|^2 + (1 - a_n) (1 - s) \|y_n - p\|^2
\]

\[
\leq a_n \|x_n - p\|^2 + (1 - a_n) (1 - s) \|x_n - p\|^2
\]

So, \( \langle x_n \rangle \) is bounded sequence so, \( \langle f_n \rangle \) also bounded

Since, \( \langle x_n \rangle \) is bounded sequence then there exists a subsequence \( \langle x_{nk} \rangle \) of \( \langle x_n \rangle \) such that \( x_{nk} \rightarrow z \)

\[
\|x_n - f_n x_n\| = \|a_n f_{nk} x_{nk} + b_n T_n x_n + c_n P_c T_n y_n\|
\]

\[
\leq a_n \|f_{nk} x_{nk} - x_n\| + (1 - a_n) \|T_{nk} f_{nk} y_{nk} - f_n x_n\|
\]

\[
\leq a_n \|f_{nk} x_{nk} - x_n\| + b_n \|T_{nk} f_{nk} y_{nk} - f_n x_n\|
\]

\[
\|x_n - f_n x_n\| \rightarrow 0 \text{ as } n \rightarrow \infty
\]

Then \( \langle x_n \rangle \) has converges weakly to an asymptotic common fixed point of \( f_n, \forall n \in N \). ■

**Theorem (2.3):**

Let \( \langle f_n \rangle \) be a bounded sequence of \( szl \)– widering mappings belong to \( F^* \) and \( \langle T_n \rangle \) be a sequence of non-expansive mapping. Define the proximal point scheme \( \langle x_n \rangle \) as:

\[
x_{n+1} = a_n f_n x_n + b_n T_n x_n + c_n P_c T_n y_n
\]

\[
y_n = (1 - c_n) x_n + c_n P_c (x_n - w_n h x_n)
\]

Where \( \langle a_n \rangle, \langle b_n \rangle, \langle c_n \rangle \) are sequences in \((0,1)\) converges to 0 such that \( a_n \geq c_n \) but \( \langle w_n \rangle \) be a sequence in \((1,\infty)\) not converges to 0 and satisfy \( w_n < 2c_n \). If

i. \( VI(C, h) \cap (\cap_{n=1}^{\infty} Fix(T_n)) \cap \cap_{n=1}^{\infty} Fix(f_n) \) \( \neq \emptyset \), \( \Sigma c_n = \infty \)

ii. \( a_n + b_n + c_n = 1 \)

Then the proximal point scheme \( \langle x_n \rangle \) converges strongly to \( z \) such that \( z \in VI(C, h) \cap (\cap_{n=1}^{\infty} Fix(P_c T_n)) \).

**Proof:**

Let \( p \in VI(C, h) \cap (\cap_{n=1}^{\infty} Fix(T_n)) \cap \cap_{n=1}^{\infty} Fix(f_n) \)

By theorem(2.1) we get \( \langle x_n \rangle \) has converges weakly to an asymptotic fixed point Now,

\[
\|y_{n+1} - y_n\| = \|(1 - c_{n+1}) x_{n+1} + c_{n+1} P_c (x_{n+1} - w_{n+1} h x_{n+1}) - (1 - c_n) x_n
\]

\[
- c_n P_c (x_n - w_n h x_n)\| y_{n+1} - y_n\|
\]

\[
\leq (1 - c_n) \|x_{n+1} - x_n\| + \|(1 - c_n) - (1 - c_{n+1})\| \|x_{n+1}\|
\]

\[
+ c_n \|P_c (x_{n+1} - w_{n+1} h x_{n+1}) - P_c (x_n - w_n h x_n)\|
\]

\[
+ l_{n+1} \|P_c (x_{n+1} - w_{n+1} h x_{n+1})\|
\]

\[
\|y_{n+1} - y_n\| \leq \|(1 - c_n)\|x_{n+1} - x_n\| + c_n \|x_{n+1} - x_n\| + c_n \|w_{n+1} h x_{n+1} - w_n h x_n\|
\]

\[+ l_{n+1} \|P_c (x_{n+1} - w_{n+1} h x_{n+1})\|
\]

\[
\|y_{n+1} - y_n\| \leq \|x_{n+1} - x_n\| + c_n \|w_n h x_n - w_{n+1} h x_{n+1}\| + l_{n+1} \|x_{n+1}\|
\]

\[+ l_{n+1} \|P_c (x_{n+1} - w_{n+1} h x_{n+1})\|\]

\[
(4)
\]
Now, we have
\[ \| x_n - p \| = \| (1 - c_n) x_n + c_n P_c(x_n - w_n h x_n) - p \| \]
\[ \leq (1 - c_n) \| x_n - p \| + c_n \| P_c(x_n - w_n h x_n) - P_c(p - w_n h p) \| \]
\[ \leq (1 - c_n) \| x_n - p \| + c_n \| (x_n - w_n h x_n) - (p - w_n h p) \| \]
\[ \leq (1 - c_n) \| x_n - p \| + c_n \| x_n - p \| + c_n w_n h x_n - h p \| \]
\[ \leq \| x_n - p \| + w_n h x_n - h p \| \]
and,
\[ \| x_{n+1} - p \| \leq a_n \| x_n - p \| + b_n \| x_n - p \| + c_n \| y_n - p \| \]
\[ \| x_{n+1} - p \| \leq (a_n + b_n) \| x_n - p \| + c_n \| x_n - p \| + w_n h x_n - h p \| \]
\[ \leq \| x_n - p \| + w_n h x_n - h p \| \]
\[ \| x_n - x_{n+1} \| \to 0 \quad \text{as } n \to \infty \] and \( (w_n) \) not converges to 0. So
\[ \| h x_n - h p \| \to 0 \quad \text{as } n \to \infty \]
\[ \| P_c T_n y_n - y_n \| \leq \| (1 - c_n) x_n + c_n P_c(x_n - w_n h x_n) - P_c T_n y_n \| \]
\[ \leq (1 - c_n) \| x_n - P_c T_n y_n \| + c_n \| P_c(x_n - w_n h x_n) - P_c T_n y_n \| \]
\[ \leq (a_n + b_n) \| x_n - P_c T_n y_n \| + c_n \| (x_n - w_n h x_n) - T_n y_n \| \]
\[ \| P_c T_n y_n - y_n \| \to 0 \quad \text{as } n \to \infty \]
(8)

Since, \( \| x_n - y_n \| \leq \| x_n - P_c T_n y_n \| + \| P_c T_n y_n - y_n \| \)

By (6) and (8) we get,
\[ \| x_n - y_n \| \to 0 \quad \text{as } n \to \infty \]
(9)

Since \( \| h y_n - h x_n \| \leq \| h y_n - h p \| + \| h p - h x_n \| \)
\[ \| h y_n - h x_n \| \to 0 \quad \text{as } n \to \infty \]

Put \( y'_n = P_c(x_n - w_n h x_n) \)
\[ \| y'_n - y_n \| = \| (1 - c_n) x_n + c_n^\varepsilon (x_n - w_n, h x_n) - P_c (x_n - w_n, h x_n) \| \\
\leq (1 - c_n) \| x_n \| + |c_n - 1| \| P_c (x_n - w_n, h x_n) \| \\
\leq (a_n + b_n) (\| x_n \| + \| P_c (x_n - w_n, h x_n) \|) \\
\Rightarrow \| y'_n - y_n \| \to 0 \quad \text{as } n \to \infty \quad (10) \]

By (9) and (10) we get
\[ \| x_n - y'_n \| \to 0 \quad \text{as } n \to \infty \quad (11) \]

Since, \( (x_n) \) is bounded sequence then there exists a subsequence \( (x_{n_k}) \) of \( (x_n) \) such that \( x_{n_k} \to z \implies y_{n_k} \to z \).

Since \( \| x_n - T_n x_n \| \to 0 \quad \text{as } n \to \infty \) and by lemma (1.3)
\[ z \in \cap_{n=1}^\infty \text{Fix}(T_n) \]

To prove that \( z \in \cap_{n=1}^\infty \text{Fix}(P_c T_n) \cap VI(C, h) \).

Let \( T(x) = \begin{cases} h(x) + N_c(x) ; & x \in C \\ \emptyset ; & x \notin C \end{cases} \), \( T \) is maximal monotone.

Let \( (x, y) \in \text{gph}(T) \). So that
\[ y \in T(x) \implies y \in h(x) + N_c(x) \implies y - h x \in N_c x \]

(By definition of \( N_c x \))
\[ (x - y_n, y - h x) \geq 0 \quad , \quad y'_n \in C \quad (12) \]

Since \( y'_n = P_c (x_n - w_n, h x_n) \) and by definition of metric projection we get
\[ (x - y'_n, y'_n - (x_n - w_n, h x_n)) \geq 0 \]
\[ (x - y'_n, y'_n - w_n h x_n) \geq 0 \]

By (12) we get,
\[ (x - y'_{n_k}, y') \geq (x - y'_{n_k}, h x) \geq (x - y'_{n_k}, h x) - (x - y'_{n_k}, y'_n - w_n h x_n) \]
\[ = (x - y'_{n_k}, h x - w_n h x_n - y'_n + w_n h x_n) \]
\[ = (x - y'_{n_k}, h x - y'_n) + (x - y'_{n_k}, h y'_n - h x_n) - (x - y'_{n_k}, y'_n - w_n h x_n) \]
\[ \geq \varepsilon \| h x - y'_n \|^2 + (x - y'_{n_k}, h y'_n - h x_n) - (x - y'_{n_k}, y'_n - w_n h x_n) \]

By (7) & (11), we get \( (x - z, y) \geq 0 \quad \text{as } k \to \infty \)

But \( T \) is maximal monotone operator \( \implies z \in T(0) \implies z \in VI(C, h) \)

Now, to prove that \( z \in \cap_{n=1}^\infty \text{Fix}(P_c T_n) \)
If not, \( z \notin \bigcap_{n=1}^{\infty} \text{Fix}(P_{C}T_{n}) \implies z \neq P_{C}T_{n}z, \forall n \in \mathbb{N} \)

\[
\lim_{n \to \infty} \inf \|y_{nk} - z\|^2 < \lim_{n \to \infty} \inf \|y_{nk} - P_{C}T_{nk}z\|^2 \\
= \lim_{n \to \infty} \inf \|y_{nk} - P_{C}T_{nk}y_{nk} + P_{C}T_{nk}y_{nk} - P_{C}T_{nk}z\|^2 \\
= \lim_{n \to \infty} \inf \|P_{C}T_{nk}y_{nk} - P_{C}T_{nk}z\|^2 \leq \lim_{n \to \infty} \inf \|T_{nk}y_{nk} - T_{nk}z\|^2 \\
\leq \lim_{n \to \infty} \inf \|y_{nk} - z\|^2
\]

Which is a contradiction therefore, we get

\( z \in \bigcap_{n=1}^{\infty} \text{Fix}(P_{C}T_{n}) \). So, \( z \in \bigcap_{n=1}^{\infty} \text{Fix}(P_{C}T_{n}) \cap VI(C, h_{n}) \)

Now, to prove that \( x_{n} \to z \)

\[
\|x_{n+1} - z\|^2 \leq a_{n}\|f_{n}x_{n} - z\|^2 + b_{n}\|T_{n}x_{n} - z\|^2 + c_{n}\|P_{C}T_{n}y_{n} - z\|^2 \\
\leq a_{n}\|f_{n}x_{n} - z\|^2 + b_{n}\|x_{n} - z\|^2 + c_{n}\|y_{n} - z\|^2
\]

Since \( a_{n} \geq c_{n} \), we get

\[
\|x_{n+1} - z\|^2 \leq b_{n}\|x_{n} - z\|^2 + a_{n}\|f_{n}x_{n} - z\|^2 + c_{n}\|y_{n} - z\|^2 \\
\leq (b_{n} + c_{n})\|x_{n} - z\|^2 + (1 - c_{n})\|y_{n} - z\|^2 \\
\leq (1 - c_{n})\|x_{n} - z\|^2 + c_{n}\|y_{n} - z\|^2 \\
\leq (1 - c_{n})\|x_{n} - z\|^2 + c_{n}\|y_{n} - z\|^2
\]

By lemma (1.3) we get, \( \langle x_{n} \rangle \) converges strongly to \( z \).

**Theorem (2.4)**

If \( \emptyset \neq C \) convex closed in \( X \) and \( \langle f_{n} \rangle \) be a sequence of nonexpansive mapping, \( \langle T_{n} \rangle \) be an \( szl \)-widering mappings, \( h \) be a \( \alpha \)-inverse strongly monotone. Define the following the proximal point scheme

\[
y_{n} = b_{n}x_{n} + (1 - b_{n})P_{C}(x_{n} - \lambda_{n}h_{n}x_{n}) \\
x_{n+1} = a_{n}f_{n}x_{n} + (1 - a_{n})T_{n}f_{n}y_{n}
\]

where \( \langle a_{n} \rangle, \langle b_{n} \rangle \), are any decreasing sequences in \( [0, 1) \) converges to \( 0 \) and \( \langle \lambda_{n} \rangle \) be a sequence in \( (0, \infty) \) such that \( \lambda_{n} \leq \alpha \)

i. \( a_{n} + b_{n} = 1, \sum \|y_{n}\| < \infty \) and \( \sum b_{n} = \infty \)

ii. \( VI(C, h) \cap \left( \bigcap_{n=1}^{\infty} \text{Fix}(T_{n}) \right) \cap \left( \bigcap_{n=1}^{\infty} \text{Fix}(f_{n}) \right) \neq \emptyset \)

Then the proximal point scheme \( \langle x_{n} \rangle \) converges strongly to the point \( z \) of such that \( z \in \bigcap_{n=1}^{\infty} \text{Fix}(P_{C}T_{n}) \cap VI(C, h) \cap \left( \bigcap_{n=1}^{\infty} \text{Fix}(T_{n}) \right) \cap \left( \bigcap_{n=1}^{\infty} \text{Fix}(f_{n}) \right) \)

**Proof**

Let \( p \in VI(C, h) \cap \left( \bigcap_{n=1}^{\infty} \text{Fix}(T_{n}) \right) \cap \left( \bigcap_{n=1}^{\infty} \text{Fix}(f_{n}) \right) \)

By theorem(2.2) we get \( \langle x_{n} \rangle \) converges weakly to an asymptotic fixed point. Now, we have

\[
\|y_{n+1} - y_{n}\| = \|b_{n+1}x_{n+1} + (1 - b_{n+1})P_{C}(x_{n+1} - \lambda_{n+1}h_{n+1}x_{n+1}) - b_{n}x_{n} - (1 - b_{n})P_{C}(x_{n} - \lambda_{n}h_{n}x_{n})\| \\
\|y_{n+1} - y_{n}\| \leq b_{n}\|x_{n+1} - x_{n}\| + (1 - b_{n})\|P_{C}(x_{n+1} - \lambda_{n+1}h_{n+1}x_{n+1}) - P_{C}(x_{n} - \lambda_{h_{n}}x_{n})\| \\
+ |b_{n+1} - b_{n}|\|x_{n+1} - x_{n}\| + |(1 - b_{n+1}) - (1 - b_{n})| \\
\|P_{C}(x_{n+1} - \lambda_{n+1}h_{n+1}x_{n+1})\| \\
\|y_{n+1} - y_{n}\| \leq b_{n}\|x_{n+1} - x_{n}\| + (1 - b_{n})\|P_{C}(x_{n+1} - \lambda_{n+1}h_{n+1}x_{n+1}) - P_{C}(x_{n} - \lambda_{h_{n}}x_{n})\| \\
\|x_{n+1} - \lambda_{n+1}h_{n+1}x_{n+1} - x_{n} + \lambda_{h_{n}}h_{n}x_{n}\|
Since \( \langle f_n \rangle \) and \( \langle P_c \rangle \) are bounded. As \( n \to \infty \), we get
\[
\| x_{n+1} - x_n \| \to 0 \quad \text{as } n \to \infty
\]
and hence
\[
\| y_{n+1} - y_n \| \to 0 \quad \text{as } n \to \infty
\]
\[
\| x_n - f_n x_n \| \leq \| x_n - x_n \| + \| x_{n+1} - f_n x_n \|
\]
\[
\| x_n - f_n x_n \| = \| x_{n+1} - x_n \| + \| x_{n+1} - f_n x_n \|
\]
\[
\| x_n - f_n x_n \| = \| x_{n+1} - x_n \| + \| x_{n+1} - f_n x_n \| + \| x_{n+1} - f_n x_n \| + \| x_{n+1} - f_n x_n \|
\]
\[
\| x_n - f_n x_n \| \to 0 \quad \text{as } n \to \infty
\]
Since \( \langle x_n \rangle \) is bounded sequence. Then there exists \( \langle x_{n_k} \rangle \) subsequence of \( \langle x_n \rangle \) such that \( x_{n_k} \to z \). Hence,
\[
z \in \bigcap_{n=1}^{\infty} \text{Fix}(f_n)
\]
In a similar way and by using lemma(1.5), we get
\[
z \in \bigcap_{n=1}^{\infty} \text{Fix}(T_n) \quad \text{and as proof of theorem (2.3). We get, } z \in \bigcap_{n=1}^{\infty} \text{Fix}(P_c T_n) \cap \text{VI}(C, h)
\]
Now, by definition of szl – widering then we get, for each \( s, t \in (0, 1) \) then there exist \( z > 0 \) (shortly we write them \( s, z \) and \( t, \) respectively) such that
\[
\| x_{n+1} - z \| = \| f_n x_n - z \| + (1 - a_n) \| T_n f_n y_n - z \|
\]
\[
\| x_{n+1} - z \| = \| f_n x_n - z \| + (1 - a_n) \| T_n f_n y_n - z \|
\]
\[
\| x_{n+1} - z \| = (1 - b_n) (1 - s) \| x_n - z \| + b_n (1 - s) \| y_n - z \|
\]
\[
\| x_{n+1} - z \| = (1 - b_n) (1 - s) \| x_n - z \| + b_n (1 - s) \| y_n - z \|
\]
By Lemma (1.2). Hence, \( \langle x_n \rangle \) converge strongly to \( z \).

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REFERENCES


